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Conf-751108--10

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THE PHYSICS OF RELATIVISTIC ELECTRON BEAMS IN RECTANGULAR AND CYLINDRICAL GEOMETRIES

L. G. Schlitt and L. P. Bradley

November 1975

This paper was prepared for presentation to
International Topical Conference on Electron Beam Research and Technology
Albuquerque, New Mexico, November 3 - 5, 1975

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THE PHYSICS OF RELATIVISTIC ELECTRON BEAMS IN RECTANGULAR AND CYLINDRICAL GEOMETRIES*

L. G. Schlitt and L. P. Bradley
Lawrence Livermore Laboratory, Livermore, CA 94550

The use of electron beams for the direct pumping of lasers for fusion applications requires the generation of large area beams in appropriate geometries. Two geometries which are of particular interest are rectangular electron beams with planar anodes and radially converging beams with cylindrical anodes. The generation of such beams requires the management of electron trajectories in a complex combination of applied and self-generated electric and magnetic fields. The beam's self-electric field limits the emitted current and the deflection of the electron in the self-magnetic field (beam pinch) limits the beam area that can be generated from a single cathode. A simple analytic model is used to derive a scaling relationship for beam pinch in both geometries of the form $V^{1/2} \frac{w}{d} = \alpha$ where V is the diode voltage, w the beam width, d the anode-cathode spacing, and α is a weak function of the geometry. Numerical calculations are presented to show the effects of nonuniform electric fields encountered in typical geometries together with supporting experimental measurements.

* This work was performed under the auspices of the U. S. Energy Research and Development Administration under Contract No. W-7405-Eng-48.

The application of electron beams to the pumping of large scale, high pressure gas laser systems requires the generation of large area relativistic electron beams possessing a high degree of spatial uniformity and low transverse energy. In particular, a single laser power amplifier module may require a beam with an area of several square meters, electron energies of a few MeV, and current densities of the order of 100 amps/cm^2. Clearly, the generation of such an electron beam involves a great many technological problems.

One of the most severe problems to be addressed is the tendency of electrons to be deflected within the diode by the beam's self-magnetic field referred to below as beam pinch. Indeed, this deflection can be so severe as to prevent electrons from reaching the anode resulting in the creation of very highly pinched electron beams. While such beams are of immense value to electron beam pellet fusion schemes, they are useless in large scale laser applications. At constant current density, the total current, self-magnetic field, and beam pinch increase with increasing diode area. Thus, beam pinch sets an upper limit on the beam area which can be produced from a single cathode. While it is possible to generate the large area beams with an array of magnetically isolated cathodes, the viability of this approach depends upon the size that an individual cathode module can attain which, in turn, is limited by beam pinch. Thus, a central issue in the application of electron beams to pumping large scale laser systems is the limitations imposed on beam formation by beam pinch.

Section A presents an analytical model of beam pinch in two geometries appropriate for laser applications. The rectangular geometry of Fig. 1(a) is suitable for the pumping of rectangular laser volumes possessing planar surfaces. The cylindrical geometry of Fig. 1 (b) employs a radially converging electron beam suitable for pumping cylindrical volumes. Both models are one dimensional and assume that the transverse dimensions of the electron beam are large compared with the anode - cathode spacing. Section B presents numerical calculations which verify that the functional form of the analytical model is appropriate even in more complex two dimensional geometries. Section C contains some experimental evidence confirming these results.

A. Analytical Model

The calculations presented for the two geometries follow the same simple form. Relativistic mechanics is used to relate the angle, Θ, at which an electron trajectory meets the anode to the particle energy, anode - cathode spacing, and th beam's self-magnetic field. Electrodynamics is then used to relate the magnetic field to the beam current and geometry and also the beam current to the applied voltage and geometry. The result in both cases is an expression for sin Θ in terms of the quantity $\frac{w}{d} V^{1/2}$ and a weak function of diode geometry, where w is the beam width, d is the anode - cathode spacing, and V the diode voltage. Since Θ is limited, a simple expression for the constraints imposed by beam pinch can be obtained. Voltages, V, are in MV.

1. Rectangular Geometry [Fig. 1(a)]

Consider an electron emitted from a cathode at $x = 0$ with velocity $U = 0$ and the cathode potential $\phi = 0$. The electron is accelerated towards the anode at $x = d$ and potential $\phi = V$ and is subjected to a uniform magnetic field $\overline{B} = -B\,\hat{z}$. This choice for $\overline{B}$ is appropriate for calculating pinch at the ends of the beam. The relativistic Hamiltonian and conjugate momentum are:

$$H = \sqrt{(c\overline{P} - q\overline{A})^2 + m^2c^4} + q\Phi$$

$$\overline{P} = \gamma m\overline{U} + q\overline{A}$$

if we take $\overline{A} = -Bx\ \hat{y}$ and let $\Phi = \Phi(x)$ then P_y and H are constants of motion and with the initial conditions

$$H = mc^2$$

$$P_y = 0$$

so that

$$mc^2 = \sqrt{c^2\gamma^2m^2U_x^{\,2} + q^2c^2B^2x^2 + m^2c^4} + q\Phi$$

or

$$1 - \frac{q\Phi}{mc^2} = \sqrt{\gamma^2\,\frac{U_x^{\,2}}{c^2} + \left(\frac{qBx}{mc}\right)^2 + 1}$$

However, $-q\Phi$ is the kinetic energy of the particle so that

$$\gamma = 1 - \frac{q\Phi}{mc^2} = \gamma(x)$$

With

$$\overline{\beta} = \frac{\overline{U}}{c}$$

we may write

$$\gamma^2 = \gamma^2\beta_x^{\,2} + \left(\frac{qB}{mc}\right)^2 x^2 + 1$$

$$\gamma^2\beta^2 = \gamma^2\beta_x^{\,2} + \left(\frac{qB}{mc}\right)^2 x^2$$

If we define

$$r_L \equiv \left|\frac{\gamma m\beta c}{qB}\right| = r_L(x) \tag{1}$$

$$1 = \frac{{\beta_x}^2}{\beta^2} + \frac{x^2}{{r_L}^2}$$

or

$$\frac{\beta_y}{\beta} = \frac{x}{r_L}$$

The angle Θ which the particle trajectory makes with the x axis is

$$\sin\Theta = \frac{\beta_y}{\beta} = \frac{x}{r_L}$$

and in particular at the anode

$$\sin\Theta = \frac{d}{r_L(x = d)} \tag{2}$$

For a rectangular beam of uniform current density j, the magnetic field (and hence Θ) is largest at the ends of the beam $y = \pm\frac{\ell}{2}$ and is given by

$$B = -\frac{j\mu_o w}{\pi}\left\{\frac{1}{4}\ln\left[1 + \left(\frac{2\ell}{w}\right)^2\right] + \frac{\ell}{w}\tan^{-1}\left(\frac{w}{2\ell}\right)\right\} \tag{3}$$

in the limit $\ell \gg w$, this reduces to

$$B \approx -\frac{j\mu_o w}{2\pi}\left[\ln\left(\frac{2\ell}{w}\right) + 1\right] \tag{4}$$

For convenience define

$$B \equiv -\frac{j\mu_o w}{2\pi}\, f\left(\frac{\ell}{w}\right) \tag{5}$$

Then equations (1), (2), and (5) yield

$$\sin \Theta = \frac{e\mu_0}{2\pi mc} \frac{j}{\gamma\beta} wd \; f(\tfrac{\ell}{w})$$

Using Friedlander's[1] relativistic expression for space charge limited current ($V \gtrsim 0.5$ MV)

$$j = \frac{2.7 \times 10^3}{d^2} (\sqrt{\gamma} - .85)^2 \tag{6}$$

$$\sin \Theta = .32 \; \frac{(\sqrt{\gamma} - .85)^2}{\gamma\beta} \; \frac{w}{d} \; f(\tfrac{\ell}{w}) \tag{7}$$

For the range $0.5 \lesssim V \lesssim 4$ MV

$$\frac{(\sqrt{\gamma} - .85)^2}{\gamma\beta} \approx 0.26 \; V^{1/2} \tag{8}$$

$$\sin \Theta = .083 \; V^{1/2} \, \frac{w}{d} \, f(\tfrac{\ell}{w}) \tag{9}$$

where $f(\frac{\ell}{w})$ is a weak function of ℓ/w so that it is reasonable to write

$$\frac{w}{d} V^{1/2} = \alpha = \frac{12 \sin \Theta_{max}}{f(\frac{\ell}{w})} \tag{10}$$

2. Cylindrical geometry [Fig. 1 (b)]

A similar calculation for a radially converging beam with $A_z = - Br \, \ell n \frac{r}{r_c}$ where the axis of the cylinder coincides with the z-axis and r_c is the cathode radius yields

$$\sin \Theta = - \frac{r_a}{r_L} \ell n \frac{r_a}{r_c} \tag{11}$$

If one half of the total diode current returns through each end of the cylindrical anode then the maximum field occurs at the edges of the beam and is

$$B = \frac{\mu_o I}{4\pi r_a} \tag{12}$$

so that

$$\sin \theta = - \frac{e\mu_o}{4\pi mc} \frac{1}{\gamma\beta} \ln \frac{r_a}{r_c}$$

Langmuir and Compton[2] give an expression for space charge limited current in a non-relativistic diode as

$$I = 1.47 \times 10^4 \frac{V^{3/2}}{\eta^2} \frac{w}{r_a} \tag{13}$$

where

$$\eta = \varepsilon - \frac{2}{5}\varepsilon^2 + \frac{11}{120}\varepsilon^3 - \frac{47}{3300}\varepsilon^4 + .00168\varepsilon^5 \ldots\ldots$$

$$\varepsilon = \ln \frac{r_a}{r_c}$$

If we assume the same functional change for a relativistic diode, then we replace $\frac{V^{3/2}}{\gamma\beta}$ by

$$\frac{1.2\ (\sqrt{\gamma} - .85)^2}{\gamma\beta} \simeq 0.31\ V^{1/2}$$

Thus

$$\sin \theta = -.27\ V^{1/2} \frac{w}{\eta^2 r_a} \varepsilon$$

With $d = r_c - r_a$,

$$\sin \Theta = .27\, V^{1/2} \frac{w}{d} \left[1 - \frac{r_c}{r_a} \right] \frac{r}{a^2} \tag{14}$$

For convenience define

$$g\left(\frac{r_a}{r_c}\right) = \left[1 - \frac{r_c}{r_a} \right] \frac{r}{a^2}$$

For $\frac{r_c}{r_a} < 10$, $g\left(\frac{r_a}{r_c}\right) \approx \left(\frac{r_a}{r_c}\right)^{1/4}$ which is again a weak function of $\frac{r_a}{r_c}$

so that

$$\frac{w}{d}\, V^{1/2} = \alpha = \frac{3.7 \sin \Theta_{max}}{g\left(\frac{r_a}{r_c}\right)} \tag{15}$$

For both rectangular and cylindrical geometries the constraints imposed by beam pinch can be expressed in the same form $\frac{w}{d} V^{1/2} = \alpha$ where α *is a weak function of diode geometry. The details are summarized in Fig. 2.* Numerical values for α are presented in Fig. 3 where the exact expressions for f and g have been used.

In general, the requirements of the laser system will determine particle energy, current density, and anode radius in the cylindrical geometry. Equation (6) or (13) can be used to determine the anode - cathode spacing or cathode radius. Then Eq. (10) or (15) can be used with the maximum permissible Θ to determine the module width w.

B. Numerical Simulations

The two dimensional diode simulation code EGUN[3] which includes the effects of the applied electric field, the beam's space charge electric field and its self-magnetic field was used to calculate electron trajectories in a more realistic rectangular geometry to determine if the functional form of the pinch scaling relationship holds in complex geometries.

The geometry chosen is shown in Fig. 4 and consists of a planar anode and a cathode whose width is comparable to the anode - cathode spacing. It is obvious that Eq. (9) cannot be used directly to calculate Θ for such a geometry since the numerical simulation leads to values of C with the opposite sign. However, it is still possible that the functional form of Eq. (10) is valid.

Figure 4 shows two calculations in which the diode voltage was held constant while the cathode dimension, emitting region, and anode - cathode spacing were all increased by a factor of 10. The electron trajectories and current density profiles are virtually identical in the two cases while the current per unit length dropped by a factor of 10 and the local current density by a factor of 100, thus scaling as d^2. Since the quantity $\frac{w}{d} V^{1/2}$ remained constant in the two cases, the similarity of electron trajectories is consistent with the analytical model.

Figure 5 presents the results of several calculations in which the cathode shape, emitting region, and anode - cathode spacing were held constant while the voltage was varied from 0.5 to 4 MV. With w taken as the width of the beam at the anode, the quantity $\frac{w}{d} V^{1/2}$ varied by only 10% for the factor of 8 change in V, which is again consistent with the analytical model.

C. Experimental Measurements

Experimental measurements of the pinch angle in rectangular geometries are presented in Figs. 6 and 7. The planar case of Fig. 6 corresponds closely to the assumptions of the analytic model. A planar anode and cathode were used together with a field shaping electrode at cathode potential which maintained a uniform electric field at the edges of the cathode. Emission from the field shaping electrode was prevented with a dielectric coating. The solid curve for this case is the prediction of the analytical model with $f(\frac{\ell}{W})$ chosen to give the correct description of $B(z)$. The other cases in this figure show that the pinch angle can be substantially reduced by shaping the cathode and anode.

Figure 7 presents the results of measurements made in a more complex geometry. The width of the cathode was less than the anode - cathode spacing, no field shaping electrode was used, and the ends of the cathode were shaped to make the emission more uniform along its length. Equation (9) predicts a value of Θ = .85 radius at y = ± 25 cm which is reasonably close to the measurements considering that the cathode was shaped at the ends.

D. Conclusions

A one dimensional analytical model which describes beam pinch and the limitations that it imposes on beam size has been presented for both rectangular and cylindrical geometries. The limitation can be expressed in the form

$$\frac{w}{d} V^{1/2} = \alpha$$

where α is a weak function of diode geometry, w is the beam width, d the anode - cathode spacing, and V the diode voltage. Numerical simulations of electron trajectories in complex two dimensional geometries indicate that this functional form remains valid. Experimental measurements support this treatment.

F. Acknowledgments

The authors are particularly indebted to J. Taska for performing the numerical calculations.

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FIGURE LEGENDS

Fig. 1 Geometries used in analytic model. a) Rectangular geometry, (b) Cylindrical geometry.

Fig. 2 Summary of important equations.

Fig. 3 Calculated values of α for rectangular and cylindrical geometries. Exact expressions for $f(\frac{\ell}{w})$ and $g\,\frac{r_a}{r_c}$ used. $\Theta_{max} = .79$

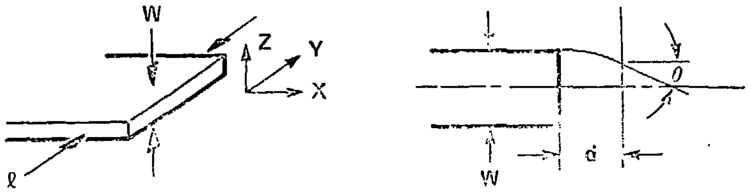
Fig. 4 Scaling of electron trajectories with geometry. Numerical simulations for conditions indicated.

Fig. 5 Scaling of beam width with voltage by numerical simulation. Geometry as in Fig. 4 with d = 1 cm.

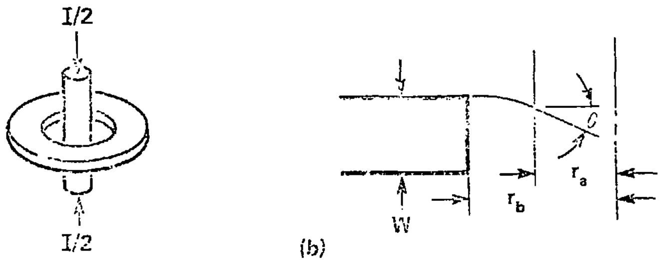
Fig. 6 Measurements of pinch angles. Planar case closely approximates analytical model. Solid line corresponds to analytical results with d = 0.8 cm, V = 1.0 MV, w = 12 cm in planar case. Shaping of anode, cathode reduces pinch angle.

Fig. 7 Measurements of pinch angles. Analytical model predicts $\Theta = .85$ at $y = \pm$ 25 cm with d = 1.5 cm, w = 3 cm, ℓ = 50 cm and V = 1.0 MV.

BEAM PINCH GEOMETRIES



(a)



(b)

Fig. 1

RECTANGULAR – CYLINDRICAL

Rectangular

$$\sin\Theta = \frac{d}{r_L}$$

$$\sin\Theta = 0.083\, V^{1/2}\, \frac{w}{d}\, f\left(\frac{\ell}{w}\right)$$

$$\frac{w}{d}\, V^{1/2} = \alpha = \frac{12 \sin\Theta_{max}}{f\left(\frac{\ell}{w}\right)}$$

$$f\left(\frac{\ell}{w}\right) = \left[\ell n\left(\frac{2\ell}{w}\right) + 1\right] \quad ; \ \ell \gg w$$

Cylindrical

$$\sin\Theta = -\frac{r_a}{r_L}\, \ell n\, \frac{r_a}{r_c}$$

$$\sin\Theta = 0.27\, V^{1/2}\, \frac{w}{d}\, g\left(\frac{r_a}{r_c}\right)$$

$$\frac{w}{d}\, V^{1/2} = \alpha = \frac{3.7 \sin\Theta_{max}}{g\left(\frac{r_a}{r_c}\right)}$$

$$g\left(\frac{r_a}{r_c}\right) \approx \left(\frac{r_a}{r_c}\right)^{1/4} \quad ; \ \frac{r_c}{r_a} < 10$$

Fig. 2

BEAM PINCH – NUMERICAL COMPARISON

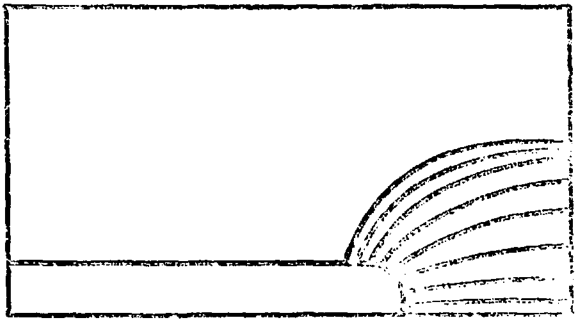
$\frac{\ell}{w}$	α
1	5.1
2	3.4
5	2.6
10	2.3
20	1.9
50	1.5
100	1.5

$\frac{r_c}{r_a}$	α
1.1	2.8
1.2	2.8
1.5	3.0
2	3.2
3	3.6
5	4.1
10	4.7

Fig. 3

GEOMETRY SCALING

V = 1.0 MV
I = 4030 amps/cm
d = 1 cm



V = 1.0 MV
I = 408 amps/cm
d = 10 cm

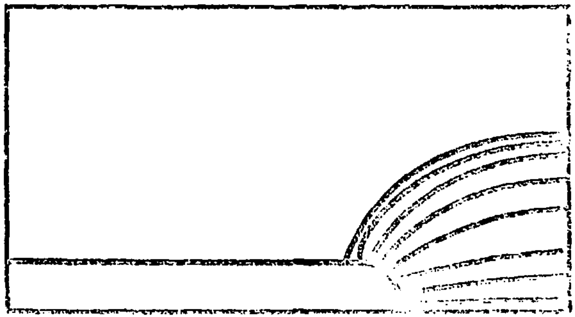


Fig. 4

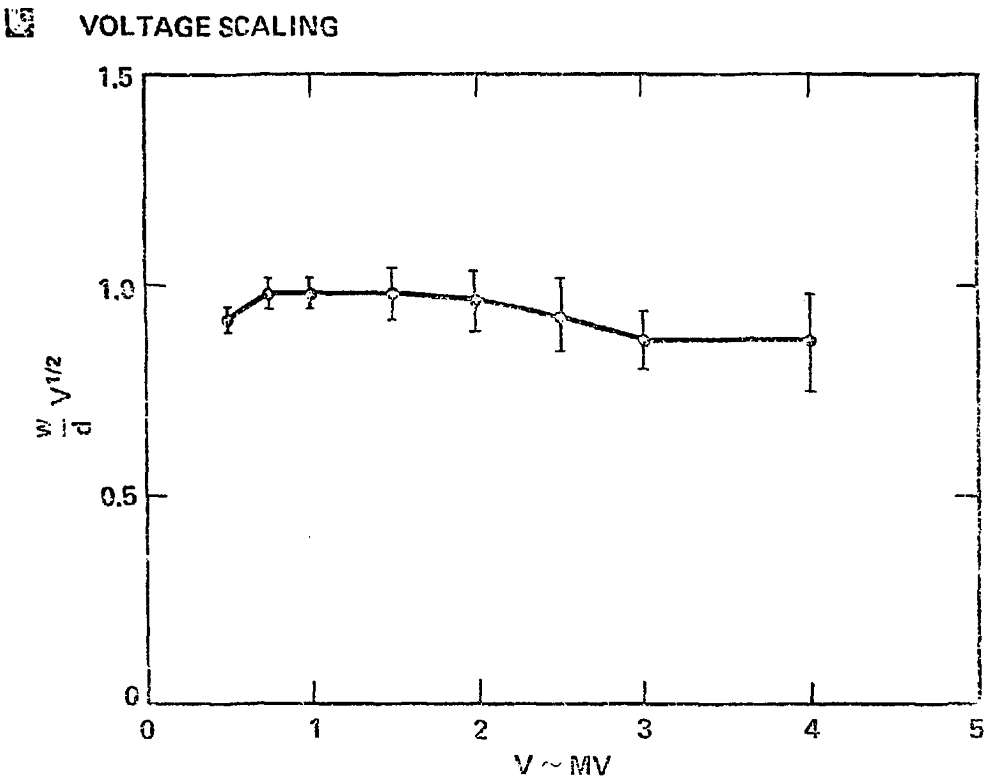


Fig. 5

SHAPED ANODE + CATHODE – MEASURED

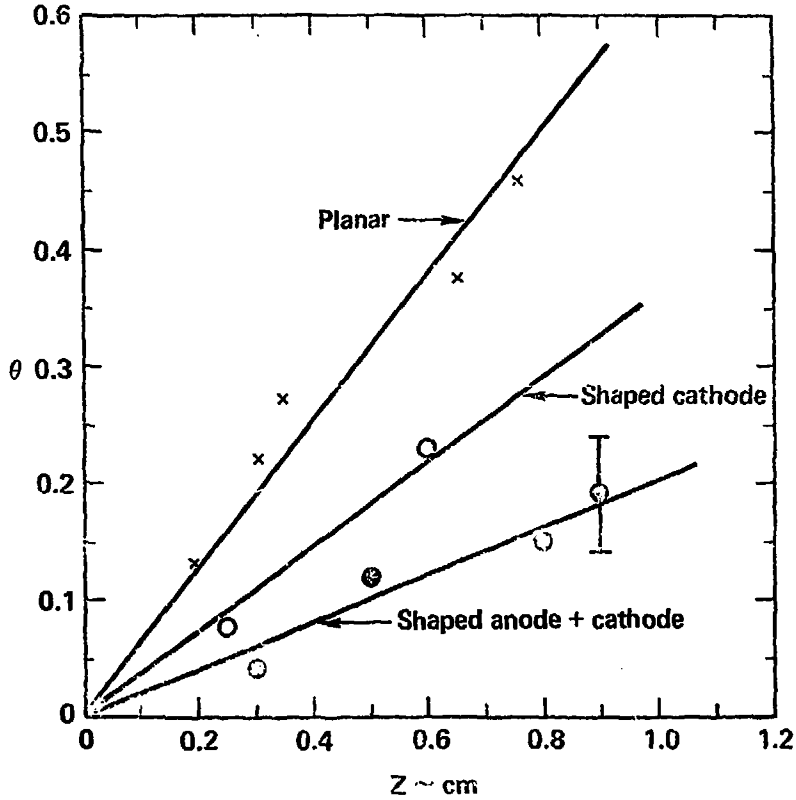
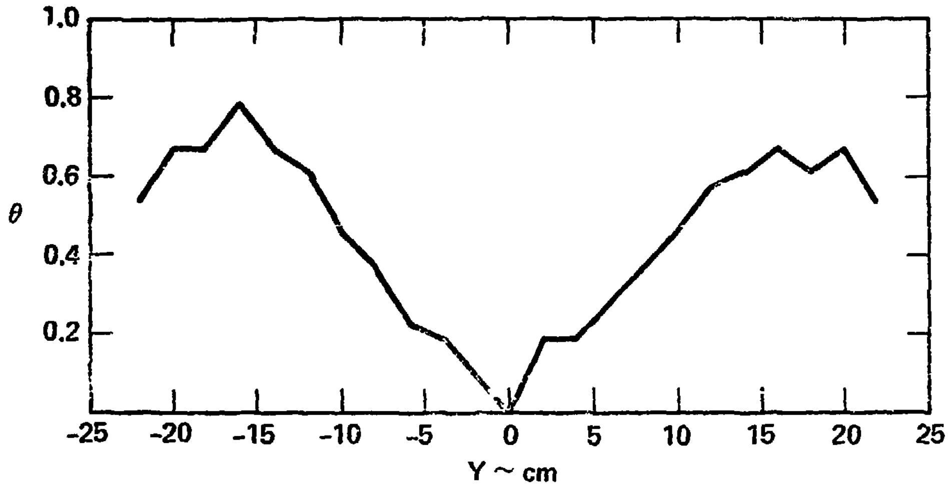


Fig. 6

BEAM PINCH-RECTANGULAR GEOMETRY



Data from Physics International Company, Felts and Stallings, PILR-675-2

Fig. 7